The parametrisation of quantisation rules equivalent to operator orderings, and the effect of

different rules on the physical spectrum

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Abstract. In this paper we recall the problem of choosing an ordering for the terms in a Hamiltonian of the form $H(p, q)$ of finite dimension in p, q. We demonstrate that most orderings which provide us with a Hermitian operator are in fact redundant and we give a general form for ordering rules in terms of arbitrary parameters. We conclude that differences in the choice of an ordering contribute differences to the physics which are only of order $\hbar^2$ and higher. We provide an example of a Hamiltonian for which the spectrum is explicitly dependent on the choice of quantisation rule (operator ordering). We compare our method of characterising operator ordering rules with that of Cohen. Finally we generalise Cohen's method to the case of all linear quantisation rules and show how his rule is a special case.

1. Introduction

The quantisation of a classical mechanical system is accomplished in general by making the following set of assignments:

$$q \rightarrow \hat{q} = q \qquad p \rightarrow \hat{p} = -i\hbar \, \partial/\partial q$$
$$t \rightarrow \hat{t} = t \qquad H \rightarrow \hat{H} = i\hbar \, \partial/\partial t. \tag{1}$$

This is the Schrödinger prescription for quantisation and is sufficient to make the link between a classical mechanical system and a quantum mechanical system for a variety of Hamiltonians, notably those of the form $H = p^2 + V(q)$. However in the more general case of Hamiltonians of the form $H = H(p, q)$ the above set of rules is not sufficient, at least not in the sense of unambiguously assigning operators to classical monomials of the form $p^i q^j$ for $i, j \geq 0$. One is faced with what has become known as the 'ordering problem', namely the problem of choosing an ordering for each of the monomials in the expression for the Hamiltonian.

Indeed it is debatable whether there is any choice of ordering which facilitates the passage from the classical case to the quantum mechanical case. It is possible that the success of the Schrödinger assignments (1) is little more than an accident. This does not appear to be the general consensus. There does appear to be a connection between classical mechanics and quantum mechanics, a connection which is nowhere more strongly suggested than by Dirac's correspondence rule, which relates the Poisson bracket of classical mechanics to the commutator bracket of quantum mechanics by the scheme

$$\{ , \} \rightarrow -\frac{i}{\hbar}[,]$$

in which the algebraic properties of the classical variables q, p and 1 are extended to the quantum mechanical operators $\hat{q}$, $\hat{p}$ and 1 by a specific choice of product.

However this may be, it is important a least to understand just how far and in what way the two theories can be deemed to 'correspond'.

The minimum requirement is that the quantisation rule should provide us with a set of Hermitian operators which give back the classical expressions in the limit of $\hbar$ going to zero.

However, it is not well understood just what freedom we have in carrying out this procedure. Many approaches are cited in the literature, taking the form of ordering rules of which the most important are:

symmetric rule $\qquad q^m p^n \rightarrow \tfrac{1}{2}(\hat{q}^m \hat{p}^n + \hat{p}^n \hat{q}^m)$

Weyl rule $\qquad q^m p^n \rightarrow \dfrac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \hat{p}^{n-k} \hat{q}^m \hat{p}^n$

Born–Jordan rule $\qquad q^m p^n \rightarrow \dfrac{1}{n+1} \sum_{k=0}^{n} \hat{p}^{n-k} \hat{q}^m \hat{p}^k.$

Of all these rules only one is distinguished from all the rest; the Weyl Rule has the property that it obeys a canonical invariance, i.e. the Weyl rule is invariant with respect to linear canonical transformations [1]. Yet, apart from this, there is no reason why any one rule should be better than any other. Before we start to tackle the problem we establish our notational conventions. We assume that the physical space has dimension d and is represented by the coordinates

$$q = (q_1, \ldots, q_d).$$

We also assume that the phase space is represented by

$$(q, p) = (q_1, \ldots, q_d, p_1, \ldots, p_d).$$

This convention is assumed in all our notation; for example, q^n is to be understood to mean

$$q_1^{n_1} \ldots q_d^{n_d}.$$

The quantisation mapping will be denoted by Ω as follows:

$$\Omega : O(q, p) \rightarrow \hat{O}(\hat{q}, \hat{p})$$

where the spaces $O(q, p)$ and $\hat{O}(\hat{q}, \hat{p})$ denote the observables of the classical system and the quantum system respectively.

2. The redundancy of operator orderings

The confusing aspect about operator orderings is that it is difficult to have a good intuition as to how they affect the physics. For instance, how do we get around the problem that two different polynomial functions of $\hat{q}$ and $\hat{p}$ might in fact be the same operator?

As an example, take the monomial qp^2. The most general operator ordering to which this can be assigned, (again within the ring generated by $\{\hat{q}, \hat{p}, 1\}$) is one of the form

$$\alpha \hat{q} \hat{p}^2 + \beta \hat{p} \hat{q} \hat{p} + \gamma \hat{p}^2 \hat{q}$$

subject to the constraint that $\alpha + \beta + \gamma = 1$. This is equivalent to requiring that the expression reduce to qp^2 in the limit as $\hbar$ tends to zero. It is usual to assume that the coefficients are real, and that the operator is Hermitian. With this in mind we find that the most general quantisation rule for the operator $\hat{q}\hat{p}^2$ is given by

$$\Omega \circ qp^2 = \alpha \hat{q}\hat{p}^2 + (1 - 2\alpha)\hat{p}\hat{q}\hat{p} + \alpha \hat{p}^2 \hat{q}. \tag{2}$$

For the values of $\alpha = \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$, we get the symmetric, the Born–Jordan and the Weyl rules respectively. However, we are at liberty to use the canonical commutation relation (CCR)

$$[\hat{q}, \hat{p}] = i\hbar. \tag{3}$$

If we apply this rule repeatedly to the operator (2), so that in the final expression all the $\hat{p}$ are on the left, and all the $\hat{q}$ are on the right then (2) becomes

$$\Omega \circ qp^2 = \hat{p}^2 \hat{q} + i\hbar \hat{p}$$

which is independent of α. It is not true to say that this is the case for all operators corresponding to monomials. However, there is still a large degree of redundancy among operator orderings. We aim to establish just what freedom there is in choosing an ordering, and just how much of a difference is made to the physics by different orderings.

A note also is in order about the choice of the CCR (3). It is generally assumed that the CCR are those given by (3). However, there has been a certain amount of speculation in the literature as to whether or not these follow from the equations of motion and, if not, what other possibilities are available. This line of questioning, it seems, was initiated by Wigner [2], and solved, at least in the simple but instructive case of the harmonic oscillator, by O'Raifeartaigh and Ryan [3] who showed that the CCR do not, in fact, follow from the equations of motion, and that other possibilities are available. However, for the purposes of this work we assume that the CCR are those given by (3).

3. A parametrisation of operator ordering rules

In quantising a monomial of the form $q^m p^n$ we get a linear combination of the different orderings of m factors of $\hat{q}$, and n factors of $\hat{p}$, subject to the requirement that the sum of the real coefficients is equal to zero (the assumption that the coefficients are real is not necessary, but seems reasonable for the moment). By using the CCR as often as necessary we can permute the factors of $\Omega \circ q^m p^n$ so as to arrive at the following result:

$$\Omega \circ q^m p^n = \hat{p}^n \hat{q}^m + i\hbar \alpha_1 \hat{p}^{n-1} \hat{q}^{m-1} + \hbar^2 \alpha_2 \hat{p}^{n-2} \hat{q}^{m-2} + i\hbar^3 \alpha_3 \hat{p}^{n-3} \hat{q}^{m-3} + \cdots \tag{4}$$

where all of the terms $\alpha_1, \alpha_2, \alpha_3$ are real and every second term contains a factor of i.

This effectively reduces the redundant degrees of freedom due to the ordering of factors (there are $\min(n+1, m+1)$ terms in this expression) yet we still require the operator to be Hermitian. To reduce the remaining degrees of freedom we must invoke the following lemma.

Lemma 1. For $\hat{q}, \hat{p}$ satisfying the relation $[\hat{q}, \hat{p}] = i\hbar$,

$$[\hat{q}^m, \hat{p}^n] = \sum_{k=1}^{\min(m,n)} i^k \hbar^k k! \binom{m}{k} \binom{n}{k} \hat{p}^{n-k} \hat{q}^{m-k}. \tag{5}$$

This lemma is mentioned by Castellani [4], and also by Wolf [5, p 142], and is easily proven either by induction or more elegantly by using the Baker–Campbell–Hausdorf formula [6]. From this lemma and the fact that

$$\hat{p}^n \hat{q}^m = \tfrac{1}{2}(\hat{p}^n \hat{q}^m + \hat{q}^m \hat{p}^n) + \tfrac{1}{2}[\hat{p}^n, \hat{q}^m] \tag{6}$$

we see that each term $\hat{p}^n \hat{q}^m$ can be written as a Hermitian part of the same degree, and other terms of lower degree. Using (5) and (6) we can replace each term in (4), transforming it into an equation of the form

$$\Omega \circ q^m p^n = \tfrac{1}{2}(\hat{q}^m \hat{p}^n + \hat{p}^n \hat{q}^m) + i\hbar \beta_1 \tfrac{1}{2}(\hat{q}^{m-1}\hat{p}^{n-1} + \hat{p}^{n-1}\hat{q}^{m-1}) + \ldots.$$

Again all of the parameters are real and every second term has a pure imaginary coefficient. We now have an expression which has been broken up into its Hermitian and anti-Hermitian parts. The constraint that the operator be Hermitian implies that the coefficients of the imaginary terms vanish. This gives the result that any ordering of the original monomial can be rewritten to display its essential degrees of freedom, taking the form

$$\Omega \circ q^m p^n = \tfrac{1}{2}(\hat{q}^m \hat{p}^n + \hat{p}^n \hat{q}^m) + \sum_{\substack{k>0 \\ m-2k \geqslant 0 \\ n-2k \geqslant 0}} \hbar^{2k} \beta_{2k} \tfrac{1}{2}(\hat{q}^{m-2k}\hat{p}^{n-2k} + \hat{p}^{n-2k}\hat{q}^{m-2k}). \tag{7}$$

Now, the number of parameters is given by $\min(\text{int}(n/2), \text{int}(m/2))$. It remains for us to determine whether any expression of the form (7) corresponds to an operator ordering, i.e. are all the α_{2k} free?

This turns out to be true, and to see why we only need to recall that the expression

$$\frac{1}{i\hbar}(\hat{q}\hat{p} - \hat{p}\hat{q}) \tag{8}$$

is equal to the identity. This means that we can take any term of the form

$$\beta_{2k} \tfrac{1}{2}(\hat{q}^{m-2k}\hat{p}^{n-2k} + \hat{p}^{n-2k}\hat{q}^{n-2k})$$

and insert $2k$ factors of the expression in (8). This provides us with a new expression equal to the original one, but now of degree n in $\hat{p}$ and degree m in $\hat{q}$. It is also easy to see that the coefficients of all the terms in the new expression sum to zero. This means that if we start off with an expression of the form (7) and insert the appropriate number of factors of the form (8), we arrive at a sum of terms, all of which contain m factors of $\hat{q}$ and n factors of $\hat{p}$ and the sum of whose coefficients is equal to one.

We now have the main result, namely that the most general quantisation rule equivalent to an ordering with real coefficients, which provides a Hermitian operator, and which gives the correct classical limit as $\hbar$ goes to zero, is given by the formula in (7) where the parameters β_{2k} are arbitrary real numbers.

As an example of the practical significance of this result, take the case of the monomial $q^4 p^5$. The most general operator ordering which corresponds to this term has 125 free parameters, before we impose Hermiticity. It is not obvious just how many degrees of freedom there really are. Our result shows that there are in fact only two free parameters. The most general operator corresponding to this monomial is given by

$$\tfrac{1}{2}(\hat{q}^4 \hat{p}^5 + \hat{p}^5 \hat{q}^4) + \hbar^2 \alpha_2 \tfrac{1}{2}(\hat{q}^2 \hat{p}^3 + \hat{p}^3 \hat{q}^2) + \hbar^4 \alpha_4 \hat{p} \qquad \alpha_2, \alpha_4 \in R.$$

From the form of (7), one can see that the choice of a specific operator ordering in a Hamiltonian is equivalent to a set of perturbations in $\hbar^2$, and that the most significant effect comes from the term of order $\hbar^2$ itself.

It is also worth noting that the only monomials to which there corresponds a unique quantisation rule are precisely the ones given by q^n, p^n, qp^n and pq^n.

These are the largest Dirac class of observables, i.e. the largest class of observables to which one can consistently apply the Dirac correspondence, or the extended Dirac correspondence as Castellani calls it [4]. The generalisation of·(7) to the case of d dimensions is straightforward.

4. The effect of different quantisation rules on the physical spectrum

An example of a specific system in which one can see the effect of different quantisation rules on the spectrum of energies is easy to contrive. We will take the case of the Hamiltonian

$$H = \tfrac{1}{2}(p^2 + q^2) + \lambda (p^2 + q^2)^3. \tag{9}$$

This may appear a rather uñlikely Hamiltonian; however, there are classical perturbation methods (for example, the Birkhof–Gustavson method of the normal form [7, 8]) which are of interest in semiclassical quantum theory and which can take the form of (9).

The term that gives the ordering problem is the second one, and the most general rule assignable to this term is given by

$$(p^2 + q^2)^3 \rightarrow \hat{p}^6 + \hat{q}^6 + \tfrac{3}{2}(\hat{p}^4\hat{q}^2 + \hat{q}^2\hat{p}^4) + 3\hbar^2\alpha\hat{p}^2 + \tfrac{3}{2}(\hat{p}^2\hat{q}^4 + \hat{q}^4\hat{p}^2) + 3\hbar^2\beta\hat{q}^2$$

where α and β are two parameters which are not necessarily equal. However, for the sake of convenience we will take them to be equal. We note that if we require the quantisation rules to be invariant with respect to the transformation $q \rightarrow -p$, $p \rightarrow q$ then this ensures that α is equal to β. (There is no other reason for doing this except that it facilitates the calculation.)

Now we can see that the most general quantisation of the Hamiltonian in (9), subject to this discrete invariance, is given by

$$\Omega \circ H = \tfrac{1}{2}(\hat{p}^2 + \hat{q}^2) + \lambda (\hat{p}^2 + \hat{q}^2)^3 + \lambda (3\hbar^2\alpha - 4)(\hat{p}^2 + \hat{q}^2).$$

The eigenfunctions of this Hamiltonian are the eigenfunctions of the Harmonic oscillator, and we see that the energy states of this system are labelled by n, and given by

$$E_n = \tfrac{1}{2}\hbar(2n + 1) + \lambda\hbar(2n + 1)^3 + \lambda\hbar(3\hbar^2\alpha - 4)(2n + 1).$$

The dependence of the spectrum on the choice of quantisation is contained in the arbitrary constant α.

5. Cohen's method of assignment

Cohen was first to introduce a method for studying different quantisation rules and their relation to the formalism of path integrals [9, 10]. This work, which evolved out of efforts to reformulate statistical mechanics, is described along with other things in the book by Langouche *et al* [11], in which they investigate the theory of functional integration from the point of view of discretisations. However, for the sake of completeness and to motivate our next result, we will outline Cohen's idea.

The idea is that a correspondence rule is given by a linear mapping from the functions on phase space, $O(q, p)$, to the operators, $\hat{O}(\hat{q}, \hat{p})$, and is completely characterised by its action on the phase-space function $\exp(iuq + ivp)$, which can be regarded as the generating function for monomials of the form $q^n p^m$. In general a correspondence rule is given by the assignment

$$\Omega \circ \exp(iuq + ivp) = \Omega(u, v) \exp(iu\hat{q} + iv\hat{p}). \tag{10}$$

By equating coefficients of u and v on either side of (10), or by operating on either side with the operator

$$\left(\frac{1}{i}\frac{\partial}{\partial u}\right)^n \left(\frac{1}{i}\frac{\partial}{\partial v}\right)^m \bigg|_{u=v=0}$$

we get the operator function of $\hat{q}, \hat{p}$ corresponding to the phase-space function of $q^n p^m$. For example, the Weyl rule, the symmetric rule and the Born–Jordan rule are obtained by simply choosing $\Omega(u, v)$ to be

$$\Omega(u, v) = 1$$

$$\Omega(u, v) = \cos\left(\tfrac{1}{2}uv\hbar\right)$$

$$\Omega(u, v) = \left(\tfrac{1}{2}uv\hbar\right)^{-1} \sin\left(\tfrac{1}{2}uv\hbar\right)$$

respectively (see [9, 4] and [11, p 18]).

It is also worth mentioning that other ordering rules which are commonly used are also included as special cases of Cohen's method, such as the normal ordering of quantum field theory [11, p 67] which is obtained using

$$\Omega(u, v) = \exp\left[\tfrac{1}{4}\hbar(u^2 + v^2)\right]$$

but differs from all the other rules in that it does not satisfy the condition implied by (11) and given in (12). The reason is that this rule is not strictly an ordering rule since it arises from a combination of a linear canonical transformation and a standard ordering in which creation operators are put to the left of annihilation operators.

Using the fact that an arbitrary $b(q, p) \in O(q, p) \cap L^2(q, p)$ can be written

$$b(q, p) = \int \tilde{b}(u, v) \exp(iuq + ivp) \, du \, dv$$

where

$$\tilde{b}(u, v) = (2\pi)^{-2} \int b(p, q) \exp(-iuq - ivp) \, dp \, dq.$$

We obtain the correspondence rule associating $\tilde{b}^{\Omega}(\hat{q}, \hat{p})$ with the phase-space function $b(q, p)$ via

$$\Omega \circ b(\hat{p}, \hat{q}) = \int \tilde{b}(u, v)\Omega(u, v) \exp(iu\hat{q} + iv\hat{p}) \, du \, dv$$

where now the freedom for choosing the correspondence rule is contained in the Ω function.

6. Implication for Cohen's assignment

By requiring that correspondence rules fulfil various criteria, for example, that they always provide the assignments

$$\Omega \circ f(q) = f(\hat{q}) \qquad \Omega \circ f(p) = f(\hat{p}) \tag{11}$$

we effectively restrict the form of the Ω function. The above conditions imply that

$$\Omega(u, 0) = 1 \qquad \Omega(0, v) = 1. \tag{12}$$

It has been remarked by Langouche *et al* [11, p 40], and also by Castellani [4], that requiring the correspondence rules for monomials like $p^n q^m$ to be equivalent to the weighted average of different orderings of n factors of $\hat{p}$ and m factors of $\hat{q}$ means that the Ω function depends on u and v through the function $uv\hbar$, i.e. that

$$\Omega(u, v) = 1 + \alpha(uv\hbar) + \beta(uv\hbar)^2 + \dots.$$

However, in the light of our results one can now say that if our correspondence rules are equivalent to real orderings, and furnish Hermitian operators, then we have

$$\Omega(u, v) = \omega(u, v) \cos(\tfrac{1}{2} uv\hbar)$$

where

$$\omega(u, v) = 1 + \alpha'(uv\hbar)^2 + \beta'(uv\hbar)^4 + \dots \tag{13}$$

and all of the constants α' and β' are free parameters.

To see that this is so consider the following:

$$\Omega(u, v) \exp(iu\hat{q} + iv\hat{p}) = \omega(u, v) \cos(\tfrac{1}{2} uv\hbar) \exp(iu\hat{q} + iv\hat{p})$$
$$= \omega(u, v)\tfrac{1}{2}(\exp(iu\hat{q}) \exp(iv\hat{q}) + \exp(iv\hat{p}) \exp(iu\hat{q})) \tag{14}$$

in which the term

$$\tfrac{1}{2}(\exp(iu\hat{q}) \exp(iv\hat{p}) + \exp(iv\hat{p}) \exp(iu\hat{q}))$$

can be regarded as the generating function for terms of the form

$$\tfrac{1}{2}(\hat{q}^n \hat{p}^m + \hat{p}^m \hat{q}^n).$$

If we take the expression in (14) and consider its Taylor expansion in terms of u and v we only have to look at the coefficient of $i^{n+m} u^n v^m$ to see to what function of $\hat{q}$ and $\hat{p}$ the monomial $q^n p^m$ has been mapped. It is easy to compare this expression with the one in (7) to see that $\omega(u, v)$ must take the form we have given it in (13).

From one point of view this is very nice, since (13) places a powerful restriction on the form that the Ω function must take. However, the preceding discussion also serves to illustrate the fact that Cohen's method of assignment is not able to furnish us with all the correspondence rules compatible with our requirements, and shows furthermore that this method is quite restrictive in the sense that once we determine the $O(\hbar^2)$ correction for one operator then the $O(\hbar^2)$ correction is determined for all the other operators.

As an example, there is no ω function which makes the following set of assignments:

$$p^3 q^3 \rightarrow \tfrac{1}{2}(\hat{q}^3 \hat{p}^3 + \hat{p}^3 \hat{q}^3) + \tfrac{1}{2}\alpha\hbar^2(\hat{p}\hat{q} + \hat{q}\hat{p})$$
$$p^4 q^4 \rightarrow \tfrac{1}{2}(\hat{q}^4 \hat{p}^4 + \hat{p}^4 \hat{q}^4) + \tfrac{1}{2}\beta\hbar(\hat{p}^2\hat{q}^2 + \hat{q}^2\hat{p}^2) + \gamma$$

for arbitrary choices of α and β. In this case Cohen's method gives $\alpha = 4\beta$. Similar relations hold for all the other monomials and the corrections to all other orders, as can be seen by considering the power-series expansion of (14).

7. Generalisation of Cohen's method

As a basis for the spaces of classical and quantum observables, we use the following sets of functions labelled by the continuous parameters $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$:

$$\exp(iuq + ivp) \qquad \exp(iu\hat{q} + iv\hat{p}).$$

If we require that our quantisation rules be linear then the effect of Ω on the base functions of $O(q, p)$ determines the quantisation of an arbitrary $f(q, p) \in O(q, p)$ as follows:

$$\Omega \circ \exp(iuq + ivp) = \int \Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \, ds \, dt. \tag{15}$$

The $\Omega(u, v, s, t)$ are a family of generalised functions and contain complete information about the quantisation rules.

We are now in a position to write down the most general linear quantisation rule relating these two spaces of observables:

$$\Omega \circ f(q, p) = \Omega \circ \int \tilde{f}(u, v) \exp(iuq + ivp) \, du \, dv$$

$$= \int \tilde{f}(u, v)(\Omega \circ \exp(iuq + ivp)) \, du \, dv$$

$$= \int \tilde{f}(u, v)\Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \, du \, dv \, ds \, dt$$

where

$$\tilde{f}(u, v) = \frac{1}{(2\pi)^{2d}} \int f(q, p) \exp(-iuq - ivp) \, dq \, dp.$$

So far we have only required linearity of the quantisation rules and there has been no restriction on the form that $\Omega(u, v, s, t)$ might take.

If we require that our rules provide Hermitian operators then we have

$$\Omega \circ f(q, p) = (\Omega \circ f(q, p))^*$$

where * denotes complex conjugation. This implies that

$$\int \tilde{f}(u, v)\Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \, du \, dv \, ds \, dt$$

$$= \int \tilde{f}(u, v)^*\Omega(u, v, s, t)^* \exp(-is\hat{q} - it\hat{p}) \, du \, dv \, ds \, dt$$

$$= \int \tilde{f}(-u, -v)^*\Omega(-u, -v, -s, -t)^* \exp(is\hat{q} + is\hat{p}) \, du \, dv \, ds \, dt$$

$$= \int \tilde{f}(u, v)\Omega(-u, -v, -s, -t)^* \exp(is\hat{q} + is\hat{p}) \, du \, dv \, ds \, dt.$$

This tells us that the requirement of Hermiticity places the following restriction on the form of Ω:

$$\Omega(u, v, s, t) = \Omega(-u, -v, -s, -t)^*. \tag{16}$$

This is quite similar to Cohen's result in that, instead of (15), Cohen had

$$\Omega \circ \exp(iuq + ivp) = \omega(u, v) \exp(iu\hat{q} + iv\hat{p})$$

and in his case the requirement of Hermiticity means that, instead of (16), the following has to hold:

$$\omega(u, v) = \omega(-u, -v)^*.$$

The requirement that we retrieve the classical limit is easily seen to be

$$\lim_{\hbar \to 0} \Omega(u, v, s, t) = \delta(u - s, v - t)$$

compared with the requirement in Cohen's scheme that

$$\lim_{\hbar \to 0} \omega(u, v) = 1.$$

8. Relationship of the general rule to the Cohen rule

To clarify the relationship between these two schemes it is helpful to consider a choice for $\Omega(u, v, s, t)$ of the form

$$\Omega(u, v, s, t) = \sum_{n,m \geqslant 0} i^{n+m} \frac{u^n v^m}{n! m!} \varphi_{nm}(s, t).$$

Since we are dealing with linear quantisation rules we can look at the Taylor expansion of both sides of (15) and compare powers of u and v. This provides us with the mappings between the monomials induced by the quantisation mapping Ω. We find that the classical monomials are mapped according to

$$\Omega \circ q^n p^m = \int \Phi_{nm}(s, t) \exp(is\hat{q} + it\hat{p}) \, ds \, dt.$$

It is by now obvious that the functions Φ_{nm} given by

$$\Phi_{nm}(s, t) = \omega_{nm}(s, t) \frac{1}{(2\pi)^{2d}} \int q^n p^m \exp(-isq - itp) \, dq \, dp$$

provide us with the freedom to quantise each one of the classical monomials in any way we like. Each ω_{nm} plays exactly the same role as the ω functions of Cohen, and we can see by inspection that, given the special form of $\Omega(u, v, s, t)$, the requirement of Hermiticity in the new scheme reduces to that in the old scheme.

9. The condition of translational invariance

One of the remarkable features of the quantisation rules of Cohen is that they are translationally invariant [11, 12]. We will now show that the requirement of translational invariance reduces the more general method of algebraic quantisation to the original method of Cohen. Before discussing this we introduce some more notation.

Define $\tau(\alpha, \beta)$ to be the translation operator whose action on both $O(q, p)$ and $\hat{O}(\hat{q}, \hat{p})$ is given by

$$\tau: \tau(\alpha, \beta) \circ f(q, p) = f(q+\alpha, p+\beta)$$

$$\tau: \tau(\alpha, \beta) \circ f(\hat{q}, \hat{p}) = f(\hat{q}+\alpha, \hat{p}+\beta).$$

Then the requirement of translational invariance is the requirement that

$$\Omega \circ \tau(\alpha, \beta) \circ f(q, p) = \tau(\alpha, \beta) \circ \Omega \circ f(q, p) \qquad (17)$$

for all $f(q, p) \in O(q, p)$. This is seen to be satisfied trivially by Cohen's quantisation rules. Applying (17) to the case of an arbitrary quantisation rule we get

$$\Omega \circ \tau(\alpha, \beta) \circ f(q, p)$$

$$= \Omega \circ \frac{1}{(2\pi)^{2d}} \int \tilde{f}(u, v) \exp(iuq + ivp) \exp(iu\alpha + iv\beta) \, du \, dv$$

$$= \frac{1}{(2\pi)^{2d}} \int \tilde{f}(u, v) \Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \exp(iu\alpha + iv\beta) \, du \, dv \, ds \, dt$$

$$= \tau(\alpha, \beta) \circ \Omega \circ f(q, p)$$

$$= \tau(\alpha, \beta) \circ \frac{1}{(2\pi)^{2d}} \int \tilde{f}(u, v) \Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \, du \, dv \, ds \, dt$$

$$= \frac{1}{(2\pi)^{2d}} \int \tilde{f}(u, v) \Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \exp(is\alpha + it\beta) \, du \, dv \, ds \, dt.$$

This tells us that

$$\int \tilde{f}(u, v) \Omega(u, v, s, t) \, (\exp(iu\alpha + iv\beta)$$

$$- \exp(is\alpha + it\beta) \exp(is\hat{q} + it\hat{p})) \, du \, dv \, ds \, dt = 0$$

and we want this to hold for all $\alpha, \beta, \hat{q}, \hat{p}$ and for all $f(q, p) \in O(q, p)$. The requirement that it holds for all $f(q, p) \in O(q, p)$ means that

$$\int \Omega(u, v, s, t) \, (\exp(iu\alpha + iv\beta) - \exp(is\alpha + it\beta) \exp(is\hat{q} + it\hat{p})) \, ds \, dt = 0.$$

That this condition holds for all $\hat{q}, \hat{p}$ requires that

$$\Omega(u, v, s, t) \, (\exp(iu\alpha + iv\beta) - \exp(is\alpha + it\beta)) = 0 \qquad (18)$$

at least in the sense of distributions. Again this must vanish for all α and β.

The only possibilities are that either $\Omega(u, v, s, t)$ vanishes everywhere or, on the points where it does not vanish, it forces the term $\exp(iu\alpha + iv\beta) - \exp(is\alpha + it\beta)$ to vanish instead. In other words, the only solution to (18) is

$$\Omega(u, v, s, t) = \omega(u, v) \delta(u - s, v - t).$$

If we substitute this expression for $\Omega(u, v, s, t)$ in (15) then we arrive at the following result:

$$\Omega \circ \exp(iuq + ivp) = \int \Omega(u, v, s, t) \exp(is\hat{q} + it\hat{p}) \, ds \, dt$$

$$= \int \omega(u, v) \delta(u - s, v - t) \exp(is\hat{q} + it\hat{p}) \, ds \, dt$$

$$= \omega(u, v) \exp(iu\hat{q} + iv\hat{p}).$$

We have established that the only linear quantisation rules which have the property of translational invariance are those of Cohen.

As a final remark on the translational invariance property of quantisation rules, it is worth pointing out that this is equivalent to the condition that the quantisation mapping Ω commute with ∂_i, the partial derivatives. The latter are well defined and unambiguous for arbitrary functions of $\hat{q}, \hat{p}$.

10. Conclusion

What we find interesting about all this is that nature seems capable of telling us more about quantisation rules. We no longer have to deal with the huge number of parameters involved in characterising operator ordering rules, and so this information seems more accessible. Despite this, different choices of ordering lead to differences in the physics of $O(\hbar^2)$ and higher. This of course corresponds to the case of real ordering rules (all of the usual rules fall into this category); however, since the operators $\hat{q}, \hat{p}$ are generally regarded as the generators of a ring over C, it might be more appropriate to consider the more general case of complex orderings. In this case the differences to the physics will be of $O(\hbar)$ and higher.

There is a terminology useful in the context of § 2, namely that of enveloping algebras. The enveloping algebra for any given Lie algebra is the quotient algebra obtained by considering the free algebra modulo the commutation relations. Effectively what we have done in § 3 is to construct the enveloping algebra generated by $\{\hat{q}, \hat{p}, \hbar\}$ modulo $[\hat{q}, \hat{p}] = i\hbar$. The problem of quantisation is therefore the problem of finding a map Ω from $O(q, p)$ into the enveloping algebra.

It can be regarded as either comforting or disturbing that the $O(\hbar^k)$ differences in the quantisation of observables depend on a single parameter in any translationally invariant linear quantisation scheme. The requirement of translational invariance is natural in the sense that translations are the simplest possible canonical transformations. This property (as defined in § 9) is, however, coordinate dependent, and a Hamiltonian in spherical coordinates does not generally have the property of invariance under a translation of r, θ and ϕ. For this method of dealing with quantisation to have any real value we have to be able to investigate systems in different coordinate representations and how our quantisation rules transform between them. The Cohen rules do not constitute the only class of linear quantisation rule, and for the reasons already mentioned we should be prepared to deal with schemes which do not fall within the Cohen's framework. This is not to say that his scheme is not useful (after all, the conventional rules are all amenable to his method of assignment) but it should, however, be borne in mind that any results proven within this framework do not treat the problem in all generality.

The non-commutativity of $\hat{q}$ and $\hat{p}$ is not easy to work with and we have only dealt with the case where the commutation relations are of the simple form in (3). O'Raifeartaigh and Ryan [3] have shown that even in the case of the simple harmonic oscillator the CCR need not be of that form. Many other operator algebras which do not have such simple relations are used in physics, and it would be of interest to investigate the effect of different orderings on the quantisation of systems in other representations, such as the angular momentum representation.

Another area where we might exploit this parametrisation of quantisation rules is that where semiclassical methods are employed, i.e. in cases where both classical and

quantum mechanical canonical transformations are used (we will refer to these as CCT and QCT respectively). It is well known that CCT and QCT do not in general commute with the order of quantisation [5, p 226] and one of the difficulties in studying this area is that when we make a transformation from one system to another we do not always understand how the quantisation rules transform. However, it might be possible in systems of reasonable complexity that, rather than worrying about the validity of interchanging CCT and QCT for a fixed method of quantisation, we can use the freedom of choosing different quantisation rules to find classes of systems for which the CCT and QCT can be made to commute. We hope to have more to say on this topic in another paper still in preparation.

Acknowledgments

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